# THE EQUATIONS FOR INTRODUCTORY ASTRONOMY

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## 1. Confusion!!!!

Science is a game of measurement. Every number has, unlike pure math, something attached to it. The things that we attach to any number in science are called "units." You can drive a car on the highway at 80 miles per hour. A high school track star runs a mile in four minutes. When you make a cake, you start with 4 cups of flour. In each of these, the number is not just a pure number, but rather it measures a quantity of some real thing. You can have four cars, four miles and four hours. They are all fours, but they are quite different fours. While you can always add or subtract numbers in the most abstract sense, the real world tells you that there is no meaning at all to the question "What is four cars plus four hours minus four miles?" However, there can be a meaning to the measurement of "four cars per mile." Perhaps we are trying to measure the average usage of a road for a county road-repair planning committee. Someone could have measured over many days the total number of cars that pass along a road at various locations. This could be averaged over many days to give this useful number.

If we take a huge number of measurements of different aspects of the same types of things then, in addition to numbers with units attached to them, we might find relationships between different kinds of measurements. To riff on the above example, you can measure the speeds of cars on many different roads. You might find that cars go faster on one kind of road as opposed to another: (i.e. highways versus back county roads). This example shows a pretty basic relationship, but if you work for a city and want to help people get around more efficiently and get rid of traffic jams, then you'll do studies like these. Now, if we look at all of science, many such relationships have been discovered. Some are useful or profound. But Nature has a way of telling us those relationships. Nature always uses the language of mathematics to demonstrate the relationships. This in itself is extremely interesting, that Nature's Laws can be best expressed mathematically, and if we are lucky, in the form of a simple equation. Sometimes, the relationship is so complicated, that it's better to use a graph to show it. And sometimes the graphs are so complicated that we run big computer simulations and make movies out of them. All of these are valid ways of speaking mathematically. Let's keep it easy, and just start with equations. They are the first steps, but they cause the most confusion, because they are long thoughts compressed into a shorthand, compact form.

Equations have letters, numbers, symbols and implied operations. Each of these four objects have distinct meanings, and they can only play with each other in specific ways. For the uninitiated, it is often painfully confusing to know when a letter in an equation is a variable, and when it is a unit. Even worse, sometimes a unit is a combination of a letter or set of letters and an exponent. Fortunately, any equation can be read like a long and complicated sentence. Once you decode the equation into a sentence, then you cannot go wrong. Here is how you do it.

Start by reading equations as sentences. That is what they are, so you might as well begin by reading them like that. Once you get familiar with them, their meanings will become second nature. It takes time, but you can do it. One of Newton's Laws of Motion is simply: $F = m \cdot a$. At first glance, we might say "F is m times a." You don't need a Doctorate to know that that sentence has no meaning. We could say "Force is mass times acceleration." That's better, but still not very enlightening. Now let's try "A force is defined to be the product of a mass and its acceleration." This is now a complete thought. There are other ways to phrase it, to be sure, but this works well to understand a problem or to tell you that one measurement is related to another measurement to create a new thing, called a force, which we will presumably use later.

Once you read an equation to know what it is, you can then plug in the appropriate numbers and units. For our purposes, a force is measured in "Newtons", mass in "kilograms", and acceleration in "meters per second per second", then the equation which originally read as a definition can now be read as a question. Here is how.

If we wrote $m \cdot a = F$, it could then be read "If I push a one kilogram mass such that it accelerates at one meter per second per second, then I'm pushing on it with a force of one Newton." Now let's do it again, but now let's move the bits of the equation around. Let's make it look like this: $F/m = a$. This now reads: "If I push a mass of one kilogram with a force of 1 Newton, then the mass accelerates at one meter per second per second." We can even do the following: $F/a = m$. "If I push a mass with a force of 1 Newton and it accelerates at one meter per second per second, then it has a mass of one kilogram."

The next step is taking the equation and substituting the numbers with their correct units into the equation and evaluating. In all of these, I've used the most basic numbers and units. In the $F/m = a$ example, we would replace "F" with 1 Newton and "m" with 1 kilogram. Then the equation would look like this:

$$\frac{1 \text{ Newton}}{1 \text{ kilogram}} = 1 \text{ meters per second per second}$$

Now, at the risk of causing confusion, we need to again condense it down to shorthand. So, we use their commonly known abbreviations.

$$\frac{1 \text{ N}}{1 \text{ kg}} = 1 \text{ m per s per s}$$

Those "per"s are really irritating so let's do this instead.

$$1 \ \frac{\text{N}}{\text{kg}} = 1 \text{ m/s/s}$$

That's still a bit clumsy, so let's do this:

$$1 \ \frac{\text{N}}{\text{kg}} = 1 \text{ m/s}^2 = 1 \ \frac{\text{m}}{\text{s}^2}$$

Finally, we can end with a special way of writing it all:

$$1 \text{ N} \cdot \text{kg}^{-1} = 1 \text{ m} \cdot \text{s}^{-2}$$

Notice that anything in the denominator of a fraction gets a negative exponent. Also notice that you can square and cube and do whatever you need to units. OK, here's where you have to be careful. We started with a general equation, and we plugged in specific numbers. We just used ones, because we wanted to focus on the units themselves. The numbers just behave like any numbers. You add, multiply, divide and subtract as you need. We can move units around from one side of the equation to the other to get the original definition.

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m} \cdot \text{s}^{-2}$$

We would read the above as the definition of the Newton. "A force of one Newton is defined to be 1 kilogram accelerated at one meter per second squared." We can really loosen up the language by saying "a Newton is a kilogram meter per second squared." You can see how this last loose way of phrasing it seems to wipe out the meaning. Notice, that from the beginning I chose the number one as our number. If we had other numbers then we would have to remember our second-grade multiplication tables. That would just add numbers, and hide the nature of the units and how they relate to each other. I could have easily have said "If I push a 5 kilogram mass such that it accelerates at 9.8 meters per second per second, then I am exerting 49 Newtons of force on the mass." See how the numbers fit in? They must have a meaning attached to them, with a unit of measurement.

Now, this is not just all circular thinking. There are other units for everything. Force can be measured in "dynes". Mass can be measured in grams or solar masses. Length can be measured in yards or miles or lightyears. Time can be measured in years or picoseconds or hummingbird heartbeats. The choice of units is completely arbitrary, and governed only by convenience. Would you measure the length of time it takes to drive across the US in microseconds or fortnights? Would you measure your height in miles? Would you measure your driving speed in millimeters per decade? You could, but everyone would be irritated with you. Choosing good units reduces irritation and makes you the life of the party.

Go back to the last equation. Did you see how when I put the "kilogram" on the other side, I removed the negative exponent? One big special rule about the game of units, is you can multiply and divide any units as many times as you like. You can create all sorts of new units if you like. But you cannot add numbers with different units on them; just like you can't add 4 miles to seven pounds. However, you can actually add 4 apples to 7 oranges, so long as you convert the units to fruit. Then it becomes "4 fruits plus 7 fruits is eleven fruits." Luckily, one apple is one fruit, as one orange is one fruit. That is, it would be odd to say "one apple is 3 fruits and one orange is 5 fruits." In a similar way, we convert units in different ways all the time. We convert feet to yards, hours to minutes and years to decades. We convert Fahrenheit to Celsius, miles to kilometers and the clothing sizes at one department store to the sizes in another. All of these are not one-to-one. Temperature even converts the zero point.

The magic of science is that we measure this thing or that thing, and if we are lucky or smart (and luck favors the diligent and persistent), we might notice that this thing's measurement is related to the thing's measurement of something else. The luminosities and masses of stars can be measured. It just so happens that there is a relationship between the mass and the luminosity of a star. As another example, there is a relationship between the brightness you measure of a light and how far away you are from the light. Finally, if you count the numbers of atoms in the tiny crystals of certain rocks, you find that sometimes the atoms transform into other atoms by fission. You can use the observed rate at which this process occurs naturally to tell you the age of the rock. Science is the process of discovering relationships like these between different measurements. The profound thoughts come after you've made many relationships. You might then discover that all these relationships are really different ways of saying the same thing. That is the deep quest of science: to find the hidden patterns in nature.

## 2. Why Measurement?

It is a common refrain by many students first encountering science; that the math is hard and they don't get it. This is because many students haven't considered that there are different kinds of knowledge. First and foremost, knowledge is something you gain while you live. You have many instinctual actions that you are born with, such as the will to breathe, fear of the dark, and knowing who mother is from all the other moms. These are things that you know from your birth, and don't reflect what you've gathered. These are instinctual and inherited knowledge, written into your DNA from tens or hundreds of millions of years of evolution. Because they are there from the start, they cannot be called bad or good, they are just there. Badness or goodness for these things are only relevant insofar as they are appropriate to the world around you. All other forms of knowledge, you get along the path of your life since birth. The sciences are concerned with things that you gather while living your life that affect your thinking or behavior. We'll call these acquired ideas "knowledge."

Generally speaking, there are two types of knowledge: empiricism and revelation. The first means knowledge acquired by experience with the world using the tools that you have around you. You don't instinctually run from fire, but you learn very fast as a baby or toddler what flame is if you're touched by it. You learn that fire can hurt, and is therefore "bad" or at least "scary." Such encounters during baby years are typical of gathering knowledge about the real world. Babies have no idea about much of these new sensations and inputs from their mouths, noses, eyes, ears and fingers, so they reach out to the world using all these "instruments" to learn about what's around them. They learn about what is harmful and what is pleasurable. They associate goodness and badness with things as they encounter them. But, occasionally, the encounter transcends goodness or badness and is fascinating for its own sake. The first campfire is transfixing, and is a mix of good and bad, requiring further knowledge about how to contain the badness (burning yourself or your things) and maximize the goodness (warmth and beauty). As the baby grows up, maybe he learns that some things burn with a nice smell and other things burn with a terrible smell. Maybe some things don't actually burn in a wood fire, and others glow green or blue. If you want to manage the location of the campfire, you use a tool like a steel tong, or a long sturdy log, to push around the embers. Studying the ways of the fire gives us control over not just the fire, but over other things. When we learn how to make it grow, how to keep it going after it's gone down to embers, we can learn how to cook with it, how to make pottery, how to start and stop it with ease, how to heat a home, how to melt iron. Each process takes different amounts of flammable material, and we learn how to do this in passing. Basically, using our hands, we fashion tools that help us control the fire, and we judge its behavior by our senses. These are the central ideas of empiricism: we harness the tools at hand to learn about and use things we find in the world around us.

Revelation is a completely different form of knowledge. It is knowledge acquired not through direct interaction with the natural world, but through some spiritual means. The spiritual knowledge we indicate here is frequently in the form of sacred books and long-held oral traditions. We learn these traditions and we hear or read the words, and we associate goodness or badness with various ideas, rather than physical objects. Revelation is nearly always in the form of a story. These stories bind ideas together with metaphors. There are many revelatory stories from many cultures supporting many traditional ways and many new ways of life. However, when we discuss revelation, it is oftentimes obtained in spite of the natural world or in opposition to it, or in an attempt to escape from it. Sometimes, revelation is discovered not by renouncing the natural world, but by isolation from other people or one's home society and immersing one-self into the natural world entirely in solitude. The number of stories that can be called revelation is enormous. However, in all cases, revelation seeks to acquire knowledge that supersedes the natural world, giving "reason" or "purpose" to the seemingly arbitrary or capricious nature of what happens to people on a day-to-day basis, or across one life or across many generations. Revelation is also strongly person-centric, meaning its goals are to provide knowledge for sentient being with "souls" to "show the way." This diverges from the purpose of science. Science doesn't address the moral or ethical aspects of revelation, and really has nothing to say about it. However, when revelation begins to speak about the workings of the natural world, that's where the clash between these two forms of knowledge arises. To wit, if a revelation stated that Earth's gravity worked because the long arms of various spiritual beings hold things down onto the Earth, then it would be chatting about the natural world, and completely subject to scientific inquiry.

Unfortunately, revelation and empiricism often clash. When a revelation, which is defined to be true by its adherents, comes in conflict with an observation about the natural world, then only one of three paths can be taken. The first path is to state that the observation of the natural world is false, and that the revelation is true. This could be particularly tricky if the revelation states that gravity doesn't work if you're so high up on top of a cliff that a heaven-ward pull will spare you a fall. The second is that the adherent to the revelation realizes the falsity of the revelation and abandons it. The third is more common: to keep both. The revelatory statement is kept in the mind of the person at the same time as the data from the natural world comes in, even if they are completely contradictory. This is true of astrology. Astrology states that the heavens directly influence people, so it is a good example of what could be called "folk revelation" or "non-religious revelation", since few, if any, religions incorporates it, and some actively deride it. The influence of the stars is demanded to be true by astrology's adherents against all observations to the contrary. Astrologers desperately try to make predictions based upon their "readings" but ultimately, can't say anything other than general platitudes about the nature of human personalities or general behavior. When there is no predictive capacity, that means there is no physical mechanism that can be seen or felt or heard or touched or experienced or measured. Astrologers are not interested in *how* it works, just that it *feels* right. This means that this "revelation" about your personal nature directly clashes with the fact that no mechanism exists to demonstrate how the knowledge was obtained from the natural world, even through planets and stars actually exist in the real world. Astrology does not inquire how the planets actually move in the sky under gravity, or what it is like to be on that planet, or to know what resources might exist on that planet. Astrology does not care to learn how the stars formed, whether or not they will ever stop shining, why the sky is dark or what the Milky Way is, or how they "stay up in the sky." The only goal of astrology is to attempt to make sense of a person's experience in this natural world using spiritual means. In that, there's nothing wrong. Understanding why you do something or why things are the way they are is a strongly spiritual pursuit. However, astrology can be easily shown to lack any kind of predictive capacity and it cannot divulge the reasons for how the revelation is obtained from the stars and planets.

Empiricism has at its core, no real concern with "why" in the sense of "purpose" or "intention". It only cares about the kind of "why" whereby which physical processes best explain the observed repeating occurrences of events in Nature. It removes the concept of "volition" or "will" from the natural world, replacing it with actions and reactions that operate according to discovered rules and laws. These rules or laws are triggered if the components are present and their setup leads to an imbalance that is resolved by nature's application of those rules and laws. As an example, imagine you put a heavy rock at the edge of a cliff made of dirt and mud. Why you put it there is of no interest. Perhaps it's a convenient place to store the rock. Perhaps you didn't see the cliff. Perhaps there is a can of beans at the bottom of the cliff you cannot open and you want to use a falling rock to open it. None of these reasons matter to the rock or the dirt or the cliff or the force of gravity or the molecular binding of the dirt or the lack of cohesion of the dirt when it gets wet in the rain and then falls to crush the can of beans. The rock, the cliff, the can of beans, the force of gravity or the forces that make the dirt hard or soft all do not “care” why they were there or why they are doing it. They don’t have will or volition, so they just behave according to natural laws. A conflicting revelation might be that the dirt “became tired of holding the rock” or that the rock “wants” to reach lower ground due to its dislike of the sky. (Of course rocks don’t like the sky, they are on the ground!) These ideas ascribe thinking or will to objects that do not have it. This would be true even if the rock was in that location for a thousand years and its precarious position was remarkable and well-known. A revelation might be that if the rock fell, it would be due to the pleasure of some deity. Since, by definition, the thoughts and reasons of deities are incomprehensible to us, we should always choose to understand how the world works and manifests itself to us first in terms of physical laws. If no physical law or process can be discovered for why the rock fell, then it is more correct to explain that we don’t understand why rocks fall to the ground when the ground holding them up is dampened rather than say that a Rain God pushed the rock off the cliff. The latter is satisfactory to many, especially people that have no responsibility for the rock’s actions. The rock will have fallen and they will move on. However, if you happen to live in a place where there are a lot of rocks on the top of a lot of cliffs and people have chosen to live under those cliffs so that they don’t have to walk very far to open their cans of beans using the frequently falling rocks, then someone in that community might actually need to know the real reason why the rocks fall off the cliffs and how to control them.

Once we need to know the true reasons why something acts the way it does in the real world, then revelations about the natural world are always completely ignored. Not that they want to be ignored, mind you. People will hold on very tight to their revelations they value. In the example above, an adherent might state that everything has a soul, even a rock, so it could be thinking, but on a “level we just don’t understand.” This is an example of the circular literalism of revelation; there can be as many revelations as one wishes. There is no end to them. The adherent to “rock-thought” might state that rocks do indeed think, but we don’t understand them because we haven’t learned to “talk like a rock.” Any rebuttal to this thinking will create new reasons why rocks think and have souls. Therefore, it becomes necessary to learn a toolbox for getting away from such rhetorical traps that lead you away from your needed purpose: which is to assure that rocks only fall from the cliffs when the cans of beans are on their targets and all people are away from the falling zone. This requires seeking out predictable rules that show us how the world actually works, not how we would like it to work or how we believe it might be working on some spiritual plane of existence. In short, revelation-based explanations of the natural world are almost always “traps” that do not lead to understanding of why something works or behaves in the way it does. It’s your choice, you could either just make sure that the cliff stays dry or you could spend years trying to talk to a rock. One of these paths provides the best chance of the needed outcome.

Therefore, empiricism acts to assist people in working with the world as it presents itself to them. It acts to solve the problems that are faced in the material world. Sometimes, these problems cannot be directly experienced by an individual because they happen on time scales that are too short or too long to easily act upon. Sometimes the problems occur in places that are extremely remote or too dangerous to venture. Or the problem is one that is chosen to be encountered rather than brought upon us by nature. This is the problem of exploration. Exploration is a fascinating human experience. When one explores, it may be for a particular purpose of gathering resources for your community. It can also be just the desire to find yourself in a new place. Exploration always contains the intent to return and show what you’ve found. This is different than a spiritual journey of discovery through isolation, since the actions undertaken on the journey are completely different. The spiritual journey seeks only to experience the voyage, but not document it. The experience itself is the sum and total of the journey, and the stories which are returned are the bounty. An exploration, however, is undertaken for a completely different reason; to document the Natural world as it presents itself and to explain it using things and processes observed to occur with your senses.

By why take these explorations into the Natural world? Why do such difficult and painstaking work that is almost never fun and easy? Perhaps there is an over-arching sense of duty or responsibility that is instilled upon us by the stories we hear in our youth and adulthood. Stories we can neither verify nor test, but that feel true anyway. There is a reason that much of the scientific studies in Medieval times was done by monks and priests in monasteries. There is a good reason why the Vatican has an astronomical observatory, and it is not to perform astrology. Johannes Kepler, in studying the motions of the planets in the sky, become the first theoretical astrophysicist, but his intent was to understand the mind of God. It is not uncommon for deeply spiritual people to be intensely interested in the Natural world, wishing to understand it intimately. Perhaps some revelations compel us to understand the Great Work of Nature as a fundamental good. All great faiths wish to help guide us in the world as we live. Therefore, understanding, appreciating, mastering and valuing Nature as it truly is are always core elements to a faith worth living with.

## 3. Units

All of science has numbers. But the numbers are related to something you measure. So every number has units attached to it, such as 5 feet, or 17 minutes, or 120 Joules. Without the units, the numbers have no meaning at all. One may be the loneliest number, but the nature of "one" all by itself, with no units, has no meaning in science. Without exception, the thing that will trip you up 100% in this whole course will be units. Without exception, not all lengths are equal. If you see a length of 1.838, you should immediately ask 1.838 *whats*. Is it in meters, miles, Ångstroms, lightyears or Megaparsecs? How many of each is in each? Nearly all problems in this class will have some sort of unit conversion. Be completely sure that you know all of them. Here is a big list of all the ones worth chatting about. Each entry is a web link. It's expected that you'll be able to look up the actual numbers online or in your textbook.

There are odd rules to units. First, they tend to written down with an abbreviation that is one letter, but this is violated all the time. Next, the unit can have an exponent, just like a variable. This counts the number of times that unit is used for that number. If one unit is written next to another, then it's a "compound unit" made up of little sub-units. We combine sub-units together by "multiplying" them together and "dividing" them up and down. But we can't get rid of a unit just because it's long. If that compound unit is used a lot, then sometimes it's given its own name.

- Distances or lengths
  - meter or m
  - kilometer or km
  - Astronomical Unit or AU = 93,000,000 miles = 150,000,000 km = $1.5 \times 10^8$ km
  - Parsec or pc = 206,264.8 AU
  - Light year or ly = speed of light times one year = $9.4607 \times 10^{12}$ km
  - Megaparsec or Gpc, which is $10^6$ pc
  - Gigaparsec or nm, which is $10^9$ pc
  - nanometer or nm which is $10^{-9}$ m
  - Ångstrom or Å which is $10^{-10}$ m
  - Schwarzschild Radius or $\mathrm{R_s}$
  - Planck Length or $l_{pl} = 1.6162 \times 10^{-35}$ m
  - Hubble Length or $c/H_o = 1.364 \times 10^{26}$ m
  - Solar Radius: $\mathrm{R}_\odot = 6.957 \times 10^5$ km
- Area
  - Square meters or $\mathrm{m}^2$
  - Square kilometers or $\mathrm{km}^2$
- Volume
  - Cubic meters or $\mathrm{m}^3$
  - Cubic kilometers or $\mathrm{km}^3$

  - Cubic parcsecs or $\mathrm{pc}^3$
  - Cubic Megaparcsecs or $\mathrm{Mpc}^3$
- Density
  - Density is basically "how much stuff in a box".
  - Kilograms per cubic meter: $\mathrm{kg} \cdot \mathrm{m}^{-3}$
  - Number of stars per cubic parcsec: "stars" $\cdot\, \mathrm{pc}^{-3}$
  - Solar Masses per cubic Megaparcsec: $\mathrm{M}_\odot \cdot \mathrm{Mpc}^{-3}$
  - Number of galaxies per cubic Megaparcsec: "galaxies" $\cdot\, \mathrm{Mpc}^{-3}$
- Time
  - Second: s
  - Earth Year: yr, which is 31,622,400 s
  - Planck Time or $t_{pl} = 5.39106 \times 10^{-44}$ s
  - Hubble Time or $t_H = 4.55 \times 10^{17}$ s = 14.4 billion years
  - Frequency in units of inverse time, the Hertz, or $1/\mathrm{s} = \mathrm{s}^{-1}$
- Speed
  - kilometers per second: $\mathrm{km/s} = \mathrm{km} \cdot \mathrm{s}^{-1}$
  - meters per second: $\mathrm{m/s} = \mathrm{m} \cdot \mathrm{s}^{-1}$
  - The speed of light in a vacuum: $c = 299,792,458$ m/s
- Acceleration
  - kilometers per second per second: $\mathrm{km/s}^2 = \mathrm{km} \cdot \mathrm{s}^{-2}$
  - meters per second per second: $\mathrm{m/s}^2 = \mathrm{m} \cdot \mathrm{s}^{-2}$
  - Acceleration due to gravity at the surface of Earth or $g = 9.80665\ \mathrm{m} \cdot \mathrm{s}^{-2}$
- Mass
  - kilogram: kg
  - Solar Mass: $\mathrm{M}_\odot = 1.98855 \times 10^{30}$ kg.
  - Chandrasekhar Mass: 1.44 $\mathrm{M}_\odot$
- Force
  - Expressed in units called Newtons with the symbol "N"
  - one kilogram · meters per second per second: $\mathrm{N} = \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
- Energy
  - Joule or "J" which is a Newton-meter.
  - breaking it down to fundamentals: $\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{m} \cdot \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2 = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
  - Electron Volt or $\mathrm{eV} = 1.6 \times 10^{-19}$ J

  - Rest Mass Energy or $\mathrm{E} = \mathrm{mc}^2$. This should look familiar.
- Power
  - Watt or one Joule per second: $\mathrm{J/s} = \mathrm{J} \cdot \mathrm{s}^{-1}$
  - breaking it down to fundamentals: $\mathrm{W} = \mathrm{J} \cdot \mathrm{s}^{-1} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{s}^{-1} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
  - Solar Luminosity: $\mathrm{L}_\odot = 3.828 \times 10^{26}$ W
- Heat
  - Temperature or $T$ in units called Kelvins, denoted "K".
  - Boltzmann constant or $k = 1.38064853 \times 10^{-23}$ J/K.
  - Stephan-Boltzmann constant or $\sigma = 5.6704 \times 10^{8}\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.

## 4. Converting Between Units

Here is how we do conversions between units. Let's convert 13.8 billion years to seconds.

$$\frac{13.8 \text{ billion years} \mid 10^9 \text{ years} \mid 365.25 \text{ days} \mid 24 \text{ hours} \mid 60 \text{ minutes} \mid 60 \text{ seconds}}{\mid 1 \text{ billion years} \mid 1 \text{ year} \mid 1 \text{ day} \mid 1 \text{ hour} \mid 1 \text{ minute}} = ?$$

We multiply numbers across and divide up and down. A unit above cancels a unit below, and we're left with only seconds across the top.

$$\frac{13.8 \mid 10^9 \mid 365.25 \mid 24 \mid 60 \mid 60 \text{ seconds}}{\mid 1 \mid 1 \mid 1 \mid 1 \mid 1}$$

Now just multiply and divide:

$$\frac{13.8 \times 10^9 \times 365.25 \times 24 \times 60 \times 60}{1 \times 1 \times 1 \times 1 \times 1} \text{ seconds} = 4.35494880 \times 10^{17} \text{ seconds}$$

We were lucky in that everything in the denominator was a 1. They could be other numbers. The treatment is the same.

Now let's convert the critical density of the universe to galaxies per cubic megaparsec.

$$\frac{9.47 \times 10^{-27} \text{ kg} \mid 1 \text{ M}_\odot \mid (3.1 \times 10^{16})^3 \text{ m}^3 \mid (10^6)^3 \text{ pc}^3}{\text{m}^3 \mid 2 \times 10^{30} \text{ kg} \mid 1 \text{ (pc)}^3 \mid 1 \text{ Mpc}^3} = 1.4 \times 10^{11} \text{ M}_\odot/\text{Mpc}^3$$

The Milky Way is about $2 \times 10^{11}$ $\text{M}_\odot$, so the universe's critical density is a small Milky Way every megaparsec.

## 5. How to Read Equations and Units

Equations and units are both composed of letters, so how do you tell them apart? In the end, you'll just have to get out your vocabulary flash cards and do a bit of memorization.

But, there are always some big clues. Whenever we make an equation, let's choose to make it with "Equation Italics", like this:

$$F = \frac{mv^2}{r}$$

Each of these italicized things in the equation are called **variables**. Each variable stands for a number with a unit attached. All variables are always just one italicized letter. Now, we can read the equation as something named $F$ is equal to $m$ times $v$ times $v$ all divided by $r$. We said "times $v$" twice because of the exponent 2 above the $v$. If they have an exponent, meaning a number to the upper right of the letter, then you multiply it by that many times. As you might have gleaned, we multiply any two variables that are next to each other, and divide by any variable that's on the bottom of a ratio. If the variable is in the denominator, then you can replace it with a negative exponent like this:

$$F = mv^2r^{-1}$$

You can re-arrange the variables in many ways, so long as you keep the sense of exponent. (What's the exponent if we haven't written it out?) The last thing you want to do in any problem is plug in numbers and units. Perhaps we want the number associated with $v$ instead of $F$. Then we rearrange the variables in the equation like this:

$$v^2 = Frm^{-1} \quad \longrightarrow \quad v = (Frm^{-1})^{1/2} = \sqrt{Frm^{-1}}$$

What we mean here is that to get $v$, you multiply all the numbers $F$, $r$ and $m^{-1}$ together, then take the square root of that number.

Now units are just things attached onto numbers. We might write the following numbers:

- First notice that if $m = 15$ kg, then $m^{-1} = \frac{1}{15\ \text{kg}} = \frac{1}{15} \cdot \frac{1}{\text{kg}} = 0.0666\ \text{kg}^{-1}$
- Therefore, if $F = 5$ N, $m^{-1} = 0.0666\ \text{kg}^{-1}$, and $r = 27$ m,
- then $v = 3$ m/s

Notice that the units have the normal font, and they can be more than one letter. You might be confused by the $r = 20$ m and $m = 15$ kg. You might be asking when m is "mass" and when is m "meters"? If it's in this exact equation and in italics, then it's "mass"; when it's attached onto a measurement as a unit and in normal font, then it's meters. You have to just become familiar with them. Also, we use $m$ for magnitude. Remember that equations don't have units in them; they only have variables which represent numbers with units.

Finally, we sometimes encounter variables with subscripts, like these:

$$m_1 \cdot d_1 = m_2 \cdot d_2$$

This always means that the things with the subscripts are the same type of variable, but the subscript indicates that we're measuring the properties of two different objects. We might write a solution like this:

- If $m_1 = 5$ kg, $d_1 = 12$ m, and $m_2 = 20$ kg,

- then $d_2 = 3$ m

The $\cdot$ things you see in there stand for multiplication. If you think about it for a bit, this is the "teeter-totter equation". The big kid has to sit closer to the center of the teeter-totter so that the little kid can balance him by sitting farther away.

You've now seen that units on numbers can cause deep confusion as to which is which. Let's now think of solving physics-based math problems as different gears of a car being driven. Just like driving, if you skip a gear, the car has a tougher time going. We'll not skip gears. We'll read each thing on its own in its own way.

(1) Collect the equations you will need to get answers.

(2) Use algebra to manipulate the equations to make all input measurements on one side of the equation, and the result you are hunting for on the other.

(3) Convert all measurements to common units. The best common units are those of your universal constants. They are your guide to what units you need to use.

(4) Insert the numbers into the equations with the common units, to obtain the resulting derived measurement.

(5) Convert the derived measurement back to the required units, if necessary.

Let's do a big example. We'll go through a series of linked equations starting with the "teeter-totter" equation, as well as a bunch more equations to show us how we can learn something about a distant planet orbiting around another star, even though we cannot directly see it. First let's collect all the measurements of the star/planet system. By measuring the Doppler shift of the light from the star, we learn the star is wobbling back and forth at a speed of $V_{star} = 5$ kilometers per hour $= 1.39$ meters per second. We also learn from the Doppler shift that it wobbles around with a period of $P_{star} = 11.185$ days $= 966{,}384$ seconds. We can also look at the star's spectrum, and by knowing about many other identical stars that the star has a mass of $M_{star} = 0.123\ M_{\odot} = 2.446 \times 10^{29}$ kilograms. Now we need Newton's Law of Gravity, the force to keep something going in a circle, Newton's version of Kepler's Third Law, the "teeter-totter" equation in speed form and in distance form.

$$\begin{aligned}
Force_{gravity} &= -\frac{GM_1M_2}{R^2} \\
Force_{circle} &= -\frac{M_2v^2}{R} \\
(\text{Period})^2 &= \frac{4\pi^2}{G(M_1+M_2)}(\text{Average Distance})^3 \\
M_1 \times v_1 &= M_2 \times v_2 \\
M_1 \times d_1 &= M_2 \times d_2 \\
\text{Average Distance} &= d_1 + d_2
\end{aligned}$$

We can then assign the variables in the equations to their respective names

$$
\begin{aligned}
F_{gravity} &= -\frac{GM_{star}M_{planet}}{R^2} \\
F_{circle} &= -\frac{M_{planet}v_{planet}^2}{R} \\
P^2 &= \frac{4\pi^2}{G(M_{star}+M_{planet})}a^3 \\
M_{star} \times v_{star} &= M_{planet} \times v_{planet} \\
M_{star} \times d_{star} &= M_{planet} \times d_{planet} \\
a &= d_{star\ to\ balance\ point} + d_{planet\ to\ balance\ point}
\end{aligned}
$$

Now we can state that the first two equations can be put together. The planet orbits in a circle around the star, so we can equate $F_{gravity}$ with $F_{circle}$.

$$
\begin{aligned}
-\frac{GM_{star}M_{planet}}{R^2} &= -\frac{M_{planet}v_{planet}^2}{R} \\
P^2 &= \frac{4\pi^2}{G(M_{star}+M_{planet})}a^3 \\
M_{star} \times v_{star} &= M_{planet} \times v_{planet} \\
M_{star} \times d_{star} &= M_{planet} \times d_{planet} \\
a &= d_{star\ to\ balance\ point} + d_{planet\ to\ balance\ point}
\end{aligned}
$$

Now we do two steps at once. We note that the sum of the masses of the star and the planet is about the same as the mass of the star. So, let's just set that sum to be just $M_{star}$. Also, let's clean up and cancel out terms on both sides of the top equation.

$$
\begin{aligned}
\frac{GM_{star}}{R} &= v_{planet}^2 \\
P^2 &= \frac{4\pi^2}{GM_{star}}a^3 \\
M_{star} \times v_{star} &= M_{planet} \times v_{planet} \\
M_{star} \times d_{star} &= M_{planet} \times d_{planet} \\
a &= d_{star\ to\ balance\ point} + d_{planet\ to\ balance\ point}
\end{aligned}
$$

We also never measure the speed of the planet, so let's insert the third equation into the first.

$$
\begin{aligned}
\frac{GM_{star}}{R} &= \frac{M_{star}^2}{M_{planet}^2}v_{star}^2 \\
P^2 &= \frac{4\pi^2}{GM_{star}}a^3 \\
M_{star} \times d_{star} &= M_{planet} \times d_{planet} \\
a &= d_{star\ to\ balance\ point} + d_{planet\ to\ balance\ point}
\end{aligned}
$$

Next, we can combine the third and fourth equations above into one equation. We'll use $d_{star}$ to link them together, because it'll be the smaller of the two distances. Also, we really want to know how far the planet is from the star, rather than the other way around.

$$\frac{GM_{star}}{R} = \frac{M_{star}^2}{M_{planet}^2} v_{star}^2$$
$$P^2 = \frac{4\pi^2}{GM_{star}} a^3$$
$$a = \frac{M_{planet}}{M_{star}} d_{planet} + d_{planet}$$

At this point, we again use the idea that the mass of the planet is much less than the mass of the star. That means we can ignore the fraction in the last equation, because it's smaller than one.

$$\frac{GM_{star}}{R} = \frac{M_{star}^2}{M_{planet}^2} v_{star}^2$$
$$P^2 = \frac{4\pi^2}{GM_{star}} a^3$$
$$a = d_{planet}$$

So, now what is $R$? It is the distance between the star and the planet, which is roughly $d_{planet}$. And we can plug in the third equation into the second one at the same time.

$$\frac{GM_{star}}{d_{planet}} = \frac{M_{star}^2}{M_{planet}^2} v_{star}^2$$
$$P^2 = \frac{4\pi^2}{GM_{star}} d_{planet}^3$$

We can now re-arrange everything a bit for both equations.

$$M_{planet} = \sqrt{\frac{d_{planet} M_{star}}{G}} v_{star}$$
$$d_{planet} = \sqrt[3]{\frac{P^2 G M_{star}}{4\pi^2}}$$

Do you see what we need to do next? We need to insert the second equation into the first. It'll look like a complete mess, but so long as it's right, we don't care if it's messy.

$$M_{planet} = \sqrt{\frac{P^{2/3} G^{1/3} M_{star}^{1/3}}{4^{1/3} \pi^{2/3}} \frac{M_{star}}{G}} v_{star}$$

This was a real wreck, so let's combine the exponents where we can.

$$M_{planet} = \sqrt{\frac{P^{2/3} M_{star}^{4/3}}{2^{2/3} \pi^{2/3} G^{2/3}}} v_{star}$$

We can further clean it up by looking at those powers of two and the square root....

$$M_{planet} = \frac{P^{1/3} M_{star}^{2/3}}{2^{1/3}\pi^{1/3}G^{1/3}} v_{star}$$

But everything has a one-third power, which is a cube root. So, we can make it look very simple.

$$M_{planet} = \sqrt[3]{\frac{P M_{star}^2 v_{star}^3}{2\pi G}}$$

So what have we done? We knew a whole bunch of relationships between a some different measurable parameters. Some things were easier to measure than others, and we used two approximations to make our lives easier. So, we now have the mass of a distant planet orbiting around a distant star dependent upon three things: the period of the orbit, the mass of the star, and the speed that we see the star wobbling in the sky due to the planet's smaller mass. We can plug in our observed quantities now and see what we get. Above "G" has been Newton's Gravitational Constant, a fact of Nature.

$$\begin{aligned} v_{star} &= 1.39 \text{ meters per second} = 1.39 \text{ m/s} \\ P_{star} &= 966,384 \text{ seconds} = 966,384 \text{ s} \\ M_{star} &= 2.446 \times 10^{29} \text{ kg} \\ G &= 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \end{aligned}$$

Let's plug all this into the equation to make sure that the units work out as well as the numbers. We can multiply all the numbers out on top and on bottom, and separate them from their units. After that, we will go step by step and simplify the units.

$$\begin{aligned} M_{planet} &= \sqrt[3]{\frac{966,384 \text{ s} \times (2.446 \times 10^{29})^2 \text{ kg}^2 \times (1.39)^3 \text{ m}^3 \text{ s}^{-3}}{2\pi \times 6.674 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}}} \\ &= \sqrt[3]{\frac{966,384 \times (2.446 \times 10^{29})^2 \times (1.39)^3}{2\pi \times 6.674 \times 10^{-11}}} \times \sqrt[3]{\frac{\text{s kg}^2 \text{ m}^3 \text{ s}^{-3}}{\text{N m}^2 \text{ kg}^{-2}}} \\ &= 7.181 \times 10^{24} \sqrt[3]{\frac{\text{s kg}^2 \text{ m}^3 \text{ s}^{-3}}{\text{N m}^2 \text{ kg}^{-2}}} \end{aligned}$$

OK, this is all good, but let's focus on the unit spaghetti mess.

$$\sqrt[3]{\frac{\text{s kg}^2 \text{ m}^3 \text{ s}^{-3}}{\text{N m}^2 \text{ kg}^{-2}}} = \sqrt[3]{\frac{\text{kg}^2 \text{ m}^3 \text{ s}^{-2}}{\text{N m}^2 \text{ kg}^{-2}}} = \sqrt[3]{\frac{\text{kg}^2 \text{ m s}^{-2}}{\text{N kg}^{-2}}} = \sqrt[3]{\frac{\text{kg}^4 \text{ m s}^{-2}}{\text{N}}} = \sqrt[3]{\text{kg}^3 \times \frac{\text{kg m s}^{-2}}{\text{N}}} = \sqrt[3]{\text{kg}^3} = \text{kg}$$

The second-to-last step uses the definition of the unit of force, the Newton "N". So we have now discovered the mass of the planet orbiting the star! $M_{planet} = 7.181 \times 10^{24}$ kg. And we can see that our approximation that the mass of the planet is much less than that of the star is completely justified.

# 6. Basic Geometry

## 6.1. Length of the path around a circle.

$$C = 2\pi R$$

- $C$ is the circumference of a circle.
- $R$ is the radius of the circle.
- $\pi$ is that famous number, 3.14159265359... Note the $\pi$ is not a variable. It is a constant with no units attached.

## 6.2. Volume of a sphere.

$$V = \frac{4\pi}{3}R^3$$

- $V$ is the volume of a sphere
- $R$ is the radius of the sphere.
- $\pi$ is that famous number, 3.14159265359...

## 6.3. Surface area of a sphere.

$$A = 4\pi R^2$$

- $A$ is the surface area of a sphere
- $R$ is the radius of the sphere.
- $\pi$ is that famous number, 3.14159265359...

# 7. Measuring Angles

## 7.1. Angular Measurement.

$$\tan\theta = \frac{s}{D}$$

- $D$ is the distance to something.
- $s$ is the parallax baseline or the size of the distant object.
- $\theta$ is the parallax angle in degrees or radians.
- "tan" is the tangent function. For a right triangle, the tangent of an angle is always the length of the side across from the angle, divided by the length of the side touching the angle that's not the hypotenuse.

## 7.2. Parallax.

$$\theta = \frac{1}{d}$$

- This is only true for very small angles.
- $d$ is now the distance to something in parsecs.
- $s$ was replaced with a distance of 1 Astronomical Unit.
- $\theta$ is the parallax angle in arcseconds, because when the angle is very small, then $\tan\theta \approx \theta$
- There are 206,264.8 AU in one parsec.

## 7.3. Space Velocity.

$$v_{space} = \sqrt{v_{radial}^2 + (4.74\mu \cdot d)^2}$$

- $v_{space}$ is the true speed that the distant star is going through space in $\mathrm{km \cdot s^{-1}}$.
- $v_{radial}$ is the radial velocity that the distant star is approaching or receding in $\mathrm{km \cdot s^{-1}}$.
- $\mu$ is the proper motion in arcseconds per year.
- $d$ is the distance in parsecs.

The "4.74" bit comes from the following, as we convert arcseconds per year to kilometers per second.

| $\mu$ arcseconds | $d$ parsecs | 1 year | 1 degree | $\pi$ radians | $3.086 \times 10^{13}$ km |
|---|---|---|---|---|---|
| year | | 31622400 seconds | 3600 arcseconds | 180 degrees | 1 parsec |

$$= 4.74\ \mu \cdot d$$

### 7.4. **Angular Resolution.**

$$\theta_{radians} = 1.220\frac{\lambda_{meters}}{D} \quad \longrightarrow \quad \theta_{arcseconds} = 0.25\frac{\lambda_{\mu m}}{D}$$

- $\theta_{radians}$ is the angular resolution in radians.
- $\lambda_{meters}$ is the wavelength of light in meters.
- $D$ is the diameter of the telescope's primary mirror or lens in meters.
- The factor of 1.220 comes from the fact that images in a telescope are always fuzzy due to diffraction. This comes only from measurement.
- $\theta_{arcseconds}$ is the angular resolution in arcseconds.
- $\lambda_{\mu m} = 10^{-6}\ \lambda_{meters}$ is the wavelength of light in micro-meters.

We can convert that to arcseconds shown below. Below, we must be sure to measure the wavelength in $\mu m$, and the diameter of the telescope's primary in meters. We could have left it in meters and meters, but then you'd be carrying around a lot of exponents. This way, it's built into the equation.

$$\frac{1.22 \text{ radians} \mid \lambda \text{ m} \mid 180 \text{ degrees} \mid 1\ \mu \text{ meter} \mid 3600 \text{ arcseconds}}{\mid D \text{ meters} \mid \pi \text{ radians} \mid 10^{-6} \text{ m} \mid 1 \text{ degree}} = 0.25\tfrac{\lambda}{D} \text{ arcseconds}$$

## 8. Kepler's Laws

8.1. **Ellipses.** All planets orbit the Sun on ellipses.

- $a$ is the average distance between one focus of the ellipse and all points on the ellipse.
- $e$ is the ellipticity of the ellipse.
- $a(1-e)$ is the distance of closest approach of the ellipse to one of the foci.
- $a(1+e)$ is the farthest distance of the ellipse to one of the foci.

8.2. **For Just the Solar System.**

$$P^2 = a^3$$

- $P$ is the orbital period of a planet around the Sun in Earth-years.
- $a$ is the average distance between a planet and the Sun in Astronomical Units.

8.3. **For any orbiting bodies anywhere in Solar Units.**

$$P^2 = \frac{a^3}{M_{total}}$$

- $P$ is the orbital period of the two objects around their common center of mass in Earth-years.
- $a$ is the average distance between the two orbiting objects in Astronomical Units.
- $M_{total}$ is the total mass of the system in Solar masses: $\mathrm{M}_{\odot}$.

8.4. **For any orbiting bodies anywhere in common units.**

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} a^3$$

- $P$ is the orbital period of the two objects around their common center of mass in seconds.
- $a$ is the average distance between the two orbiting objects in meters.
- $M_1$ and $M_2$ are the masses of the two gravitating objects in kilograms.
- $G$ is Newton's Gravitational Constant: $6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$

8.5. **Average orbital speed.**

$$v = \frac{2\pi a}{P}$$

- $P$ is the orbital period, usually in seconds or years.
- $a$ is the average distance between the two objects in some convenient units, usually kilometers.
- $v$ is the average speed of the orbiting body, in kilometers per second or per year.

## 9. Newton's Laws

### 9.1. Fundamental Definition of Force.

$$F = ma$$

- $F$ is the force in Newtons.
- $m$ is the affected mass in kilograms.
- $a$ is the acceleration of the affected mass in meters per second per second ( $\mathrm{m/s^2}$ ).

### 9.2. Momentum.

$$p = mv$$

- $p$ is the momentum of the moving thing.
- $m$ is the mass of the thing that's moving, typically in kg.
- $v$ is the speed or velocity of the thing that is moving typically in m/s.

### 9.3. Angular momentum of something moving around in a circle.

$$L = rmv = rp$$

- $L$ is the angular momentum.
- $m$ is the mass of the object that's moving in a circle.
- $v$ is the speed of the object as it goes around the circle.
- $r$ is the distance the object is from the center of the circle. (i.e. the radius of the circle.)
- $p$ is the momentum at one instant in its arc around the circle.

### 9.4. Force needed to make an object move in a circle.

$$F = \frac{mv^2}{r}$$

- $F$ is the force in Newtons.
- $m$ is the mass of the object moving in a circle.
- $v$ is the speed of the object as it goes around in a circle.
- $r$ is the distance the object is from the center of the circle. (i.e. the radius of the circle.)

## 10. Gravity

### 10.1. **Force between any two masses due to Gravity.**

$$F = \frac{Gm_1m_2}{d^2}$$

- $F$ is the force in Newtons. Incidentally, this is your weight!
- $G$ is Newton's Gravitational Constant: $6.673 \times 10^{-11}$ N $\cdot$ m$^2$/kg$^2$
- $m_1$ is the mass of one object.
- $m_2$ is the mass of the other object.
- $d$ is the distance between the two objects' centers of mass.

### 10.2. **Weight on Earth.**

$$F = mg$$

- $F$ is the force in Newtons, or weight.
- $m$ is the mass of the object near the surface of Earth.
- $g$ is the acceleration due to gravity at the surface of Earth: 9.8 m/s$^2$

### 10.3. **Center of Mass.**

$$m_1 \cdot d_1 = m_2 \cdot d_2$$

- $m_1$ is the mass of one thing.
- $m_2$ is the mass of the other thing.
- $d_1$ is the distance from one thing to the center of mass.
- $d_2$ is the distance from the other thing to the center of mass.

# 11. Energy

## 11.1. Energy required to make a force move an object.

$$E = Fd = mad$$

- $E$ is the Energy in Joules.
- $F$ is force applied to the mass of the object that's being moved.
- $d$ is the distance through which the force is applied.
- $m$ is the mass of the object being accelerated.
- $a$ is the acceleration that happens to the object when the force is applied.

## 11.2. Kinetic Energy of a moving thing.

$$E = \frac{1}{2}mv^2$$

- $E$ is the Energy in Joules.
- $m$ is the mass of the object that's moving.
- $v$ is the speed of the object.

## 11.3. Energy required to lift an object against gravity near Earth's surface.

$$E = F_{gravity}H = mgH$$

- $E$ is the Energy in $J$.
- $F_{gravity}$ is force of gravity on the object that's being moved in Newtons: N.
- $H$ is the height through which the object is lifted in m.
- $m$ is the mass of the object being accelerated in kg.
- $g$ is the acceleration due to gravity at the surface of Earth, which is about $9.8\ \mathrm{m \cdot s^{-2}}$.

## 11.4. Energy required to lift an object against gravity.

$$E = \frac{GMm}{R} - \frac{GMm}{(R+H)}$$

- $E$ is the Energy in Joules : J.
- $G$ is Newton's Gravitational Constant: $6.673 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
- $m$ is the mass of the object that's being lifted in kg.
- $M$ is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- $R$ is the original height from which the object is lifted in meters.
- $H$ is the additional height that the object is lifted in meters.

11.5. **Escape speed.**

$$E_{esc} = \frac{GMm}{R} = \frac{1}{2}mv_{esc}^2 \implies v_{esc} = \sqrt{\frac{2GM}{R}}$$

You need a specific amount energy to lift or throw something to an infinite height. That means that $H$ in the previous equation becomes huge. If an object is exactly at the escape speed from a planet or star, we mean that it never falls back down; always slowing down and finally stopping at an infinite distance.

- $E_{esc}$ is the Energy required to escape the object in Joules.
- $G$ is Newton's Gravitational Constant: $6.673 \times 10^{-11}$ N $\cdot$ m$^2$/kg$^2$
- $m$ is the mass of the object that's being lifted in kg.
- $M$ is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- $R$ is the original height from which the object is lifted in meters. It can also be the size of the planet or star, if you're escaping from the surface of such an object.
- $v_{esc}$ is the speed of the object in m/s.

11.6. **Energy of a mass at rest.**

$$E = mc^2$$

- $E$ is the Energy in Joules.
- $m$ is the mass of the object.
- $c$ is the speed of light: $3 \times 10^8$ m/s
- Yes, this is Einstein's equivalence of matter and energy!

11.7. **Conservation of Energy in a Gravitational Field.**

$$E_{total} = \frac{1}{2}mv^2 - \frac{GMm}{D}$$

This is the Total Energy of a mass as it moves up or down in the gravitational field around a planet or star, such as a ball thrown upwards. It stays the same.

- $E_{total}$ is the total energy of the object in Joules.
- $m$ is the mass of little object.
- $G$ is Newton's Gravitational Constant: $6.673 \times 10^{-11}$ N $\cdot$ m$^2$/kg$^2$
- $M$ is the mass of the big thing that's being lifted against, like a planet or a star in kg.
- $D$ is the distance between the centers of mass of the two gravitating objects.
- $v$ is the speed of the little object.

# 12. Light

### 12.1. Frequency.

$$F = \frac{1}{P}$$

- $F$ is the frequency of the light in Hertz.
- $P$ is the period of oscillation in seconds.

### 12.2. Speed of Light.

$$c = \lambda \cdot \nu = \omega \cdot f$$

- $c$ is the speed of light: $3 \times 10^8$ m/s
- $\nu$ or $f$ is the frequency of light in hertz.
- $\lambda$ or $\omega$ is the wavelength of light in meters.
- Watch out, we use MANY different lengths for wavelengths in astronomy.

### 12.3. Doppler Effect and Redshift.

$$1 + z = \frac{\lambda_{observed}}{\lambda_{emitted}} = \frac{\nu_{emitted}}{\nu_{observed}} = \sqrt{\frac{1 + v/c}{1 - v/c}} \approx 1 + \frac{v}{c}$$

- $z$ is the redshift or blueshift.
- $\lambda$ is the wavelength of the observed or emitted light.
- $\nu$ is the frequency of the observed or emitted light.
- $c$ is the speed of light: $3 \times 10^8$ m/s
- $v$ is the speed relative to the observer ( for speeds much less than that of light-speed.)
- The last approximation is true only for speeds much less than light-speed.

### 12.4. Energy of a photon.

$$E = hf = \frac{hc}{\lambda}$$

- $E$ is the energy in Joules.
- $h$ is Planck's constant: $6.626 \times 10^{-34}$ J $\cdot$ s $= 4.135668 \times 10^{-15}$ eV $\cdot$ s
- $c$ is the speed of light: $3 \times 10^8$ m/s
- $\lambda$ is the wavelength of light in meters.
- $f$ is the frequency of light in Hertz.
- Joules are not the usual units for energy of photons. Usually it's eV. Wavelengths are in many different units, so you'll have to be careful.

## 13. Blackbody Radiation

### 13.1. **Wien's Law.**

$$T = \frac{b}{\lambda_{max}}$$

- $T$ is the temperature in Kelvins: K.
- $\lambda_{max}$ is the peak wavelength of blackbody emission in meters.
- $b$ is a number: $2.897773 \times 10^{-3}\ m \cdot K$

### 13.2. **Stefan-Boltzmann law.**

$$J = \sigma T^4$$

- $J$ is the power emitted per square meter: $\mathrm{W} \cdot \mathrm{m}^{-2}$
- $T$ is the temperature in Kelvins: K.
- $\sigma$ is a number: $5.6704 \times 10^8\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.

### 13.3. **Luminosity of a star.**

$$L = 4\pi R^2 \cdot J$$

- $L$ is the luminosity in Watts
- $R$ is the radius of the star in meters.
- $J$ is the power emitted per square meter: $\mathrm{W} \cdot \mathrm{m}^{-2}$
- $\pi$ is that old friend 3.14159...

### 13.4. **Brightness of a star.**

$$B = \frac{L}{4\pi d^2} = \sigma T^4 R^2 d^{-2}$$

- $B$ is the brightness at the receiver in $\mathrm{Watts/m}^2$
- $L$ is the luminosity in Watts: W.
- $\sigma$ is a number: $5.6704 \times 10^8\ \mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4}$.
- $T$ is the temperature in Kelvins.
- $R$ is the radius of the star in meters.
- $d$ is the distance to the star in meters.

## 14. Magnitudes

### 14.1. Distance Modulus Equation.

$$m - M = 5\log_{10}(d) - 5$$

- $m$ is the apparent magnitude, which is a measure of brightness.
- $M$ is the absolute magnitude, which is a measure of luminosity.
- $d$ is the distance in parsecs, which is operated on by the base-10 logarithm.
- To get this, we chose the standard star to be at $d_0 = 10$ parsecs, so that $M_0 = m_0$.

### 14.2. Brightness - Apparent Magnitude Relationship.

$$m_{star} - m_0 = -2.5\log_{10}\left(\frac{B_{star}}{B_0}\right)$$

- $m_{star}$ is the apparent magnitude of the star, which is a unit-less measurement of brightness.
- $m_0$ is the apparent magnitude of a standard reference star, like the Sun or Vega..
- $B_{star}$ is the brightness of the star measured in $\mathrm{W/m^2}$
- $B_0$ is the brightness of a standard reference star, like the Sun or Vega.

### 14.3. Luminosity - Apparent Magnitude Relationship.

$$m_{star} - m_0 = -2.5\log_{10}\left[\frac{L_{star}}{L_0}\left(\frac{d_0}{d_{star}}\right)^2\right]$$

- $m_{star}$ is the apparent magnitude of the star, which is a measure of brightness.
- $m_0$ is the apparent magnitude of the reference star.
- $L_{star}$ is the luminosity of the star.
- $L_0$ is the luminosity of the reference star.
- $d_{star}$ is the distance of the star.
- $d_0$ is the distance to the reference star.

### 14.4. Luminosity - Absolute Magnitude Relationship.

$$M_{star} - M_0 = -2.5\log_{10}\left(\frac{L_{star}}{L_0}\right)$$

- $M_{star}$ is the absolute magnitude of the star, which is a unit-less measurement of luminosity.
- $M_0$ is the absolute magnitude of the reference star.
- $L_{star}$ is the luminosity of the star measured in Watts
- $L_0$ is the luminosity of the reference star.

## 15. Mass-Luminosity relationship of stars on Main Sequence

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^{3.5}$$

- $L$ is the luminosity of the star in Solar units.
- $L_\odot$ is the luminosity of the Sun.
- $M$ is the mass of the main-sequence star in Solar units.
- $M_\odot$ is the mass of the Sun.

This is a strictly empirical relationship, seen from the data using a graph. It's not "derivable." It's also valid only for middle-mass stars. Same goes for the next section.

## 16. Main Sequence lifetime

$$t_{MS} = t_\odot \left(\frac{M}{M_\odot}\right)^{-2.5}$$

- $t_{MS}$ is the main-sequence lifetime of the star in Solar units.
- $t_\odot$ is the main-sequence lifetime of the Sun.
- $M$ is the mass of the main-sequence star in Solar units.
- $M_\odot$ is the mass of the Sun.

## 17. Average Kinetic Energy (Energy of Motion) of a Gas

$$\frac{1}{2}\, m_{ave} v_{gas}^2 = \frac{3}{2}\, kT$$

- $v_{gas}$ is the average speed of the molecules in the gas in $\mathrm{m \cdot s^{-1}}$.
- $k$ is Boltzmann's constant $= 1.38064853 \times 10^{-23}$ $\mathrm{J \cdot K}$
- $m_{ave}$ is the average mass of the atoms and molecules of the gas.
- $T$ is the average temperature of the entire gas in Kelvin.
- Note that if a gas is more than one thing, then they have different masses, and thus different speeds for the same temperature.

## 18. Hubble Law

$$v_{radial} = H_o \cdot d$$

- $v_{radial}$ is the radial velocity that the distant galaxy is approaching or receding in $\text{km} \cdot \text{s}^{-1}$.
- $d$ is the distance in Megaparsecs.
- $H_o$ is the Hubble Constant, currently thought to be 68 $\text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$.

The universe's age can be estimated from the reciprocal of the Hubble constant.

$$\frac{1}{H_o} = \frac{\text{second} \cdot \text{Mpc} \;|\; 1 \text{ year} \;|\; 10^6 \text{ parsecs} \;|\; 3.086 \times 10^{13} \text{ km}}{68 \text{ km} \;|\; 31622400 \text{ seconds} \;|\; 1 \text{ Mpc} \;|\; 1 \text{ parsec}} = 14.3 \times 10^9 \text{ years}$$